Math 245B Lecture 2 Notes

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1 Point Set Topology

1.1 Topological spaces

A metric space defines a collection of open sets. To consider spaces without a metric, we define a collection of open sets with the same properties. This yields a more general theory than the theory of metric spaces.

Definition 1.1. Let X be a set. A **topology** on X is a a collection $\mathcal{T}$ of **open** subsets of X such that

1. $\varnothing, X \in \mathcal{T}$
2. If $\mathscr{A} \subseteq T$, then $\bigcup_{U \in \mathscr{A}} U \in \mathcal{T}$
3. If $U_1, \dots, U_m \in \mathcal{T}$¡ then $\bigcap_{i=1}^m U_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a **topological space**.[1]

Definition 1.2. A subset $C \subseteq X$ is **closed** if $X \setminus C$ (denoted C^c) is open.

Example 1.1. Every metric space is a topological space.

Example 1.2. For every set X, $\mathcal{T} = \mathscr{P}(X)$ is called the **discrete topology**.

Example 1.3. For every set X, $\mathcal{T} = \{\varnothing, X\}$ is called the **trivial topology**.

Example 1.4. For every set X, $\mathcal{T} = \{U \subseteq X : U = \varnothing \text{ or } U^c \textbf{ is finite}\}$ is called the **cofinite topology**,

Example 1.5. If $(X, \mathcal{T})$ is a topological space, and $Y \subseteq X$¡ then $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is called the **relative topology** of $\mathcal{T}$ on Y.

[1] People usually just refer to X as the topological space when $\mathcal{T}$ is understood.

1.2 Closure and convergence

Let $(X, \mathcal{T})$ be a topological space.

Definition 1.3. If $Y \subseteq X$, then $V \subseteq Y$ is **relatively open (resp. closed)** in Y if $V = U \cap Y$, where U is open (resp. closed).

Definition 1.4. If $A \subseteq X$, $A^o = \bigcup\{U : U \subseteq A, U \text{ open}\}$ is the **interior** of A.

This is the largest open set contained in A.

Definition 1.5. If $A \subseteq X$, $\overline{A} = \bigcap\{C : C \supseteq A, C \text{ closed}\}$ is the **closure** of A.

This is the smallest closed set contained in A.

Definition 1.6. $A \subseteq X$ is **dense** if $\overline{A} = X$.

Definition 1.7. The **boundary** of $A \subseteq X$ is $\partial A; = \overline{A} \setminus A^o$.

Definition 1.8. $A \subseteq X$ is **nowhere dense** if $(\overline{A})^o = \varnothing$.

Definition 1.9. A **neighborhood** of $x \in X$ is any $U \in \mathcal{T}$ such that $x \in U$. A **neighborhood** of $A \subseteq X$ is any $U \in \mathcal{T}$ such that $A \subseteq U$.

Definition 1.10. A **point of closure** of A is a point $x \in X$ such that $U \cap A \neq \varnothing$ for all neighborhoods U of x.

Proposition 1.1. *$\overline{A}$ is the set of points of closure of A.*

Proof. ($\supseteq$): Let x be a point of closure and let $C \supseteq A$. We want to show $x \in C$. If instead $x \in C^c$, then C^c is a neighborhood of x disjoint from A.So x is not a point of closure, which is a contradiction.

($\subseteq$): Let x be a non-point of closure. Then there exists a neighborhood $U \ni x$ such that $U \cap A = \varnothing$. So U^c is closed, $x \notin U^c$, and $U^c \supseteq A$. Then $x \notin \overline{A}$. $\square$

Definition 1.11. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Then x_n **converges** to x in $\mathcal{T}$ (written $x_n \to x$) if for every neighborhood U of x, there exists an n_0 such that $x_n \in U$ for all $n \geq n_0$.

Remark 1.1. Here are a few caveats. Convergence does not characterize points of closure like it does for metric spaces. Also, limits of sequences are not necessarily unique in topological spaces.

1.3 Generating topologies and bases

Definition 1.12. If $\mathcal{T}_1, \mathcal{T}_2$ are two topological spaces on X, then $\mathcal{T}_2$ is **stronger** (resp. **weaker**) than $\mathcal{T}_1$ if $\mathcal{T}_2 \supseteq \mathcal{T}_1$ (resp. $\mathcal{T}_2 \subseteq T_1$).

Lemma 1.1. *Any intersection of topologies is a topology.*

Corollary 1.1. *Any $\mathcal{E} \subseteq \mathscr{P}(X)$ generates a topology $\mathcal{T}(\mathcal{E})$.*

In this case, $\mathcal{E}$ is called a **sub-base** for the topology $\mathcal{T}(\mathcal{E})$.

Remark 1.2. Any family generates a unique topology, but a topology may be generated by many different families.

Definition 1.13. A **neighborhood base** at $x \in X$ is a collection $\mathcal{N}$ of neighborhoods of x such that for all neighborhoods $U \ni x$, ther exists a $V \in \mathcal{N}$ such that $x \in V \subseteq U$. A **base** for $\mathcal{T}$ is a family which includes a neighborhood base around every point.

Proposition 1.2. *Let $\mathcal{E} \subseteq \mathcal{T}$. Then $\mathcal{E}$ is a base for $\mathcal{T}$ if and only if every nonempty $U \in \mathcal{T}$ is a union of members of $\mathcal{E}$.*

Proof. ($\Longrightarrow$): Assume $\mathcal{E}$ is a base, and let $\varnothing \neq U \in \mathcal{T}$. THen for all $x \in U$, there exists a $V_x \in \mathcal{E}$ such that $x \in V_x \subseteq U$. So $U = \bigcup_{x \in U} V_x$.

($\Longleftarrow$): Let $x \in U \in \mathcal{T}$. Then $U = \bigcup_{V \in \mathcal{E}'} V$ for some $\mathcal{E}' \subseteq \mathcal{E}$. So $x \in$ for some $V \in \mathcal{E}'$. Now $x \in V \subseteq U$. □

These two characterizations generalize the notion of open balls in a metric space.

Proposition 1.3. *If $\mathcal{E} \subseteq \mathscr{P}(X)$, then $\mathcal{E}$ is a base for some $\mathcal{T}$ if and only if*

1. *$\bigcup \mathcal{E} = X$,*

2. *For all $U, V \in \mathcal{E}$ and for all $x \in U \cap V$, there exists a $W \in \mathcal{E}$ such that $x \in E \subseteq U \cap V$.*

Proof. ($\Longrightarrow$): Try doing this direction yourself.

($\Longleftarrow$): Let $\mathcal{T} = \{V \subseteq X : \forall x \in V, \exists U \in \mathcal{E} \text{ s.t. } x \in U \subseteq V\}$. Check that $\mathcal{T}$ is a topology, and then check that $\mathcal{E}$ is a base for $\mathcal{T}$: If $V_1, V_2 \in \mathcal{T}$ and $x \in V_1 \cap V_2$, then there exist $U_1, U_2 \in \mathcal{E}$ such that $x \in U_i \subseteq V_i$ for $i = 1, 2$. By the second property, there exists a $W \in \mathcal{E}$ such that $x \in W \subseteq U_1 \cap U_2 \subseteq V_1 \cap V_2$. So $V_1 \cap V_2 \in \mathcal{T}$. Finally, $\mathcal{E} \subseteq \mathcal{T}$, and the definition of $\mathcal{T}$ means that $\mathcal{E}$ concludes a neighborhood base at every point. □

Unlike with σ-algebras, this means that it is easy to see how we generate a topology.

Corollary 1.2. *If $\mathcal{E} \subseteq \mathscr{P}(X)$, then $\mathcal{T}(\mathcal{E}) = \{\varnothing, X\} \cup \{$unions of finite intersections from $\mathcal{E}\}$.*

Proof. Just show that $\mathcal{T}(\mathcal{E})$ is a topology. □

Example 1.6. $F \subseteq \mathbb{R}^{\mathbb{R}}$. For every $m \in \mathbb{N}$, $t_1, \ldots, t_m \in \mathbb{R}$, $x_1, \ldots, x_m \in \mathbb{R}$, and $\varepsilon > 0$, define $U(t_1, \ldots, t_m, x_1, \ldots, x_m, \varepsilon) := \{f \in F : |x_i - f(t_i)| < \varepsilon \, \forall i \leq m\}$. Let E be the set of all such $U(t_1, \ldots, t_m, x_1, \ldots, x_m, \varepsilon)$. As an exercise, show that this is a base for $\mathcal{T}(\mathcal{E})$. We claim that if $(f_n)_{n \in \mathbb{N}}$ is a sequence in F, then $f_n \to f$ in $\mathcal{T}(\mathcal{E})$ iff $f_n \to f$ pointwise. Next time, we will show that this topology is not defined by a metric.